AGAT 2016, Cargèse
a point-vortex toy model

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M.P. Rast, JFP, PRE **79** (2009)
M.P. Rast, JFP, PRL **107** (2011)
M.P. Rast, JFP, P.D. Mininni, PRE **93** (2009)

Motivations

- Turbulence, transport issues, mostly Langrangian
- Toy model
- Analytics
- Cargèse

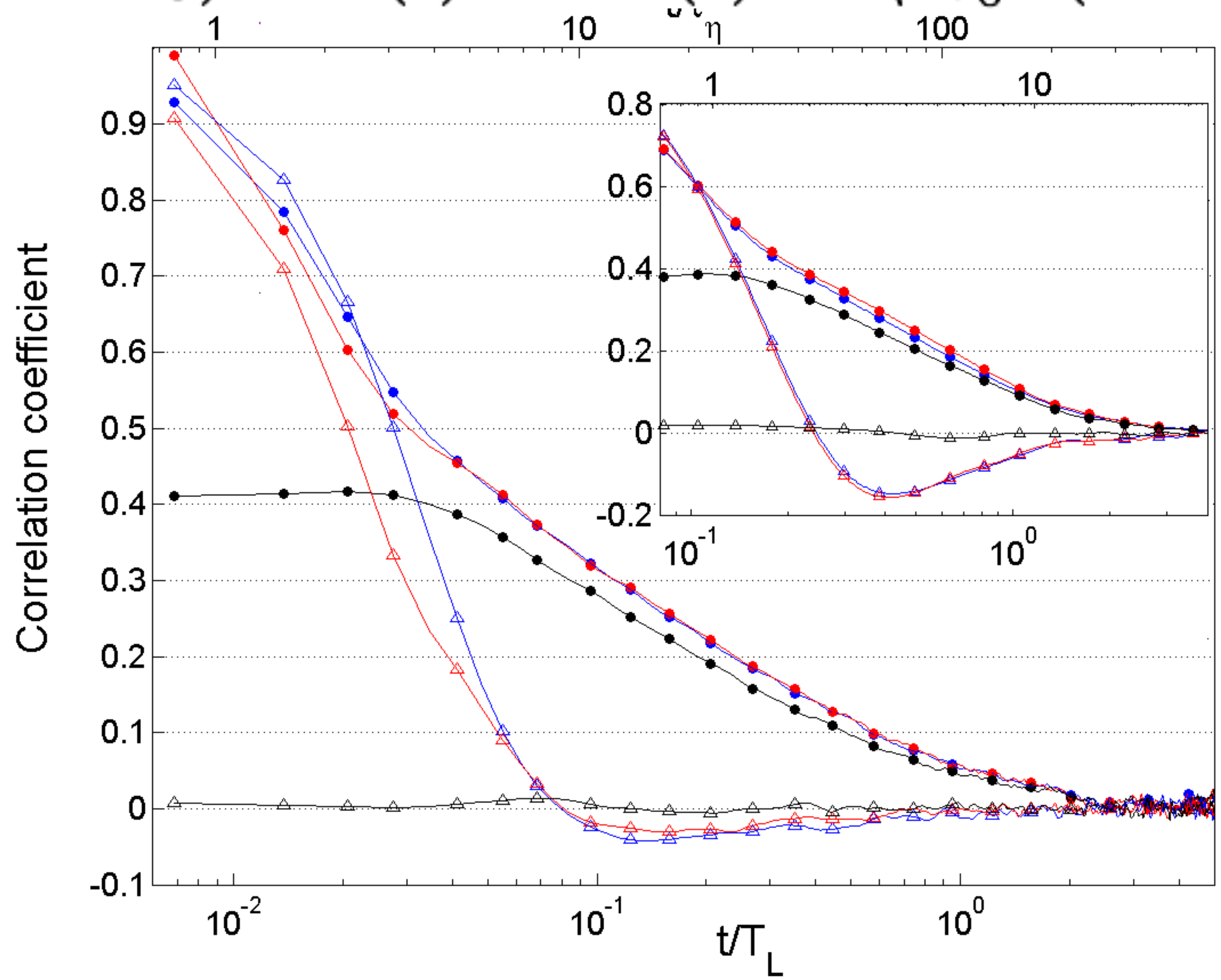


Lagrangian dynamics

$$\delta_{\tau_0} v(t) = v(t + \tau_0) - v(t) \Rightarrow C(t) = \langle \delta_{\tau_0} v(t' + t) \delta_{\tau_0} v(t') \rangle_t$$

Lagrangian dynamics

$$\delta_{\tau_0} v(t) = v(t+\tau_0) - v(t) \Rightarrow C(t) = \langle \delta_{\tau_0} v(t'+t) \delta_{\tau_0} v(t') \rangle_t$$



Mordant, Metz, Michel, Pinton, *Phys. Rev. Lett.* **89** (2002)

Lagrangian dynamics

MRW model

Bacry, Delour, Muzy, *Phys. Rev. E*, **64**, (2001).
(also Aringazin & Mazhitov, Int. J. Mod. Phys., 18 (2004))

stochastic equation for the velocity increments

$$d_t u = -\gamma(u)u + \xi(t)$$

'K41' theory : $\xi(t)$ is δ-correlated noise,

Model, from observations :

$$\xi(t) = e^{\omega(t)} G(t)$$

G(t) : gaussian , white in time, and :

$$\langle \omega(t)\omega(t+\Delta t)\rangle_t = -\lambda^2 \log(\Delta t / T_L)$$

Questions

- scalar transport without molecular diffusion

$$\partial_t c + \mathbf{u} \cdot \nabla c = S(\mathbf{x}, t)$$

- concentration, if Green's function is known

$$\langle c(\mathbf{x}, t) \rangle = \int \int P(\mathbf{x}, t | \mathbf{x}', t') S(\mathbf{x}', t') d\mathbf{x}' dt'$$

Note

DISSIPATIVE RANGE

Note : diffusion enhances separation

$$\frac{1}{2}\frac{d\langle r^2\rangle}{dt} = \frac{B\langle r^2\rangle}{\tau_\eta} + 6\kappa + 2\kappa\frac{(t-t_0)^2}{\tau_\eta^2}$$

Saffman PG. On the effect of the molecular diffusivity in turbulent diffusion. *J. Fluid Mech.* **8**:273–83 (1960)

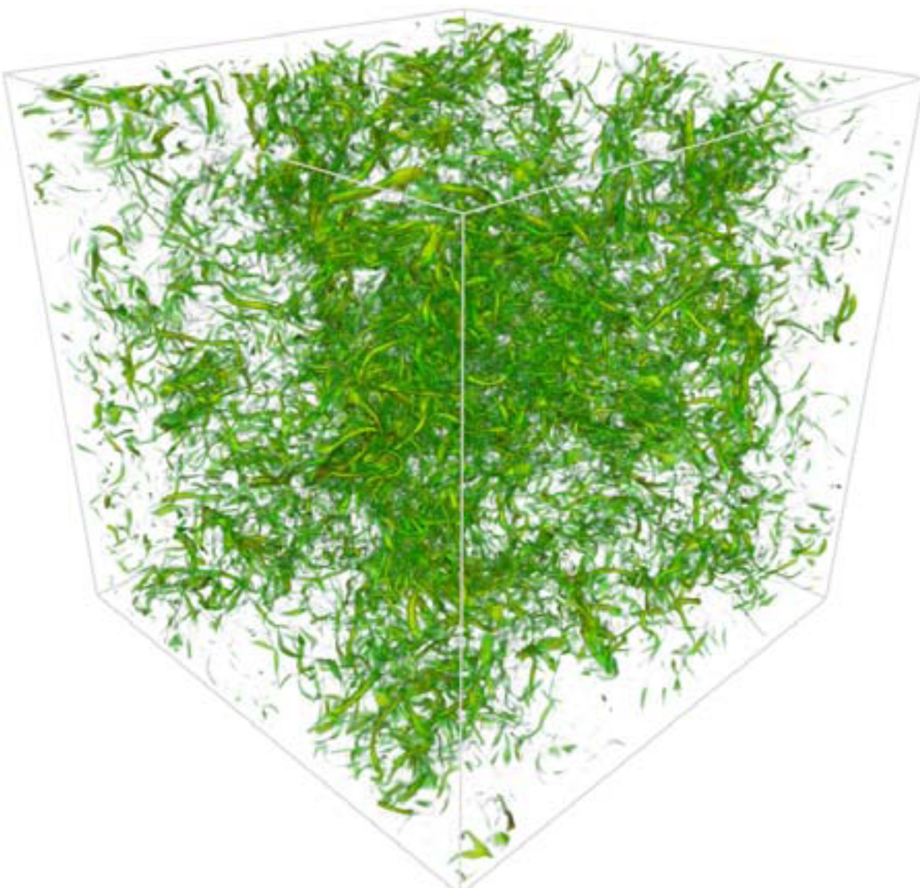
Point-vortex, as the cheapest toy model ?

- An old idea (Onsager, 1949)
- With some 3D features

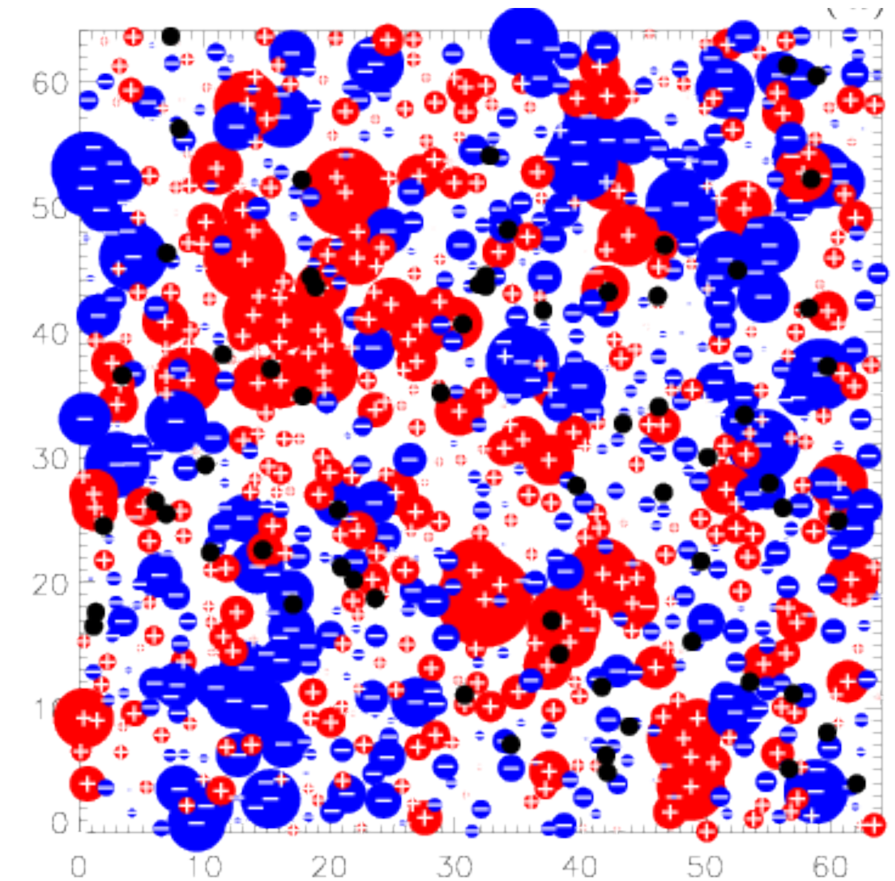
$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{1}}^{\mathbf{N}} \frac{\mathbf{\Gamma_k}}{\mathbf{2}\pi|\mathbf{x}-\mathbf{x_k}|}\left[\hat{\mathbf{z}} \times (\widehat{\mathbf{x}-\mathbf{x_k}})\right]$$

+ merger
+ injection (constant rate, Gaussian distribution)

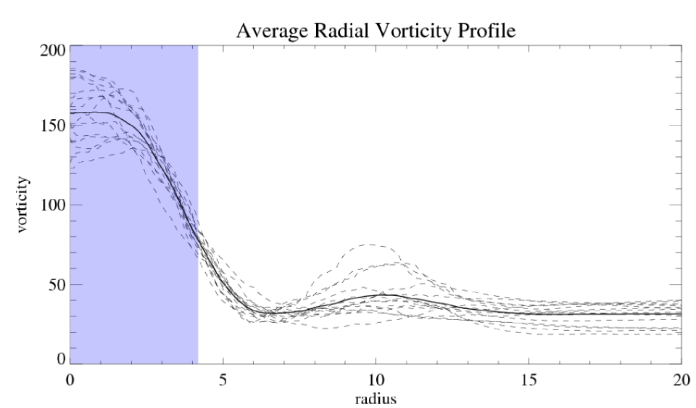
MIMIC 3D TURBULENCE WITH 2D-MODEL



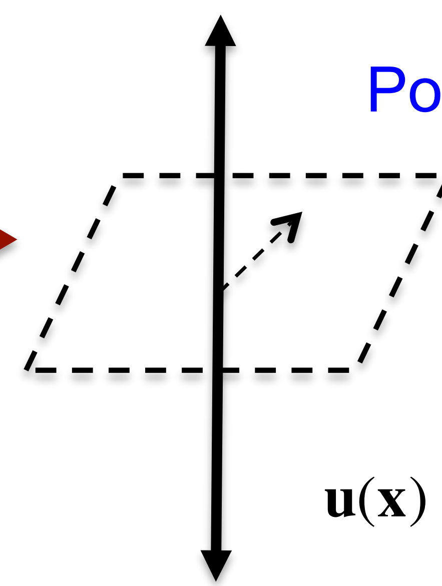
Mininni et al. (2008+)



Rast & Pinton (2009)



Gruchalla et al. (2009)

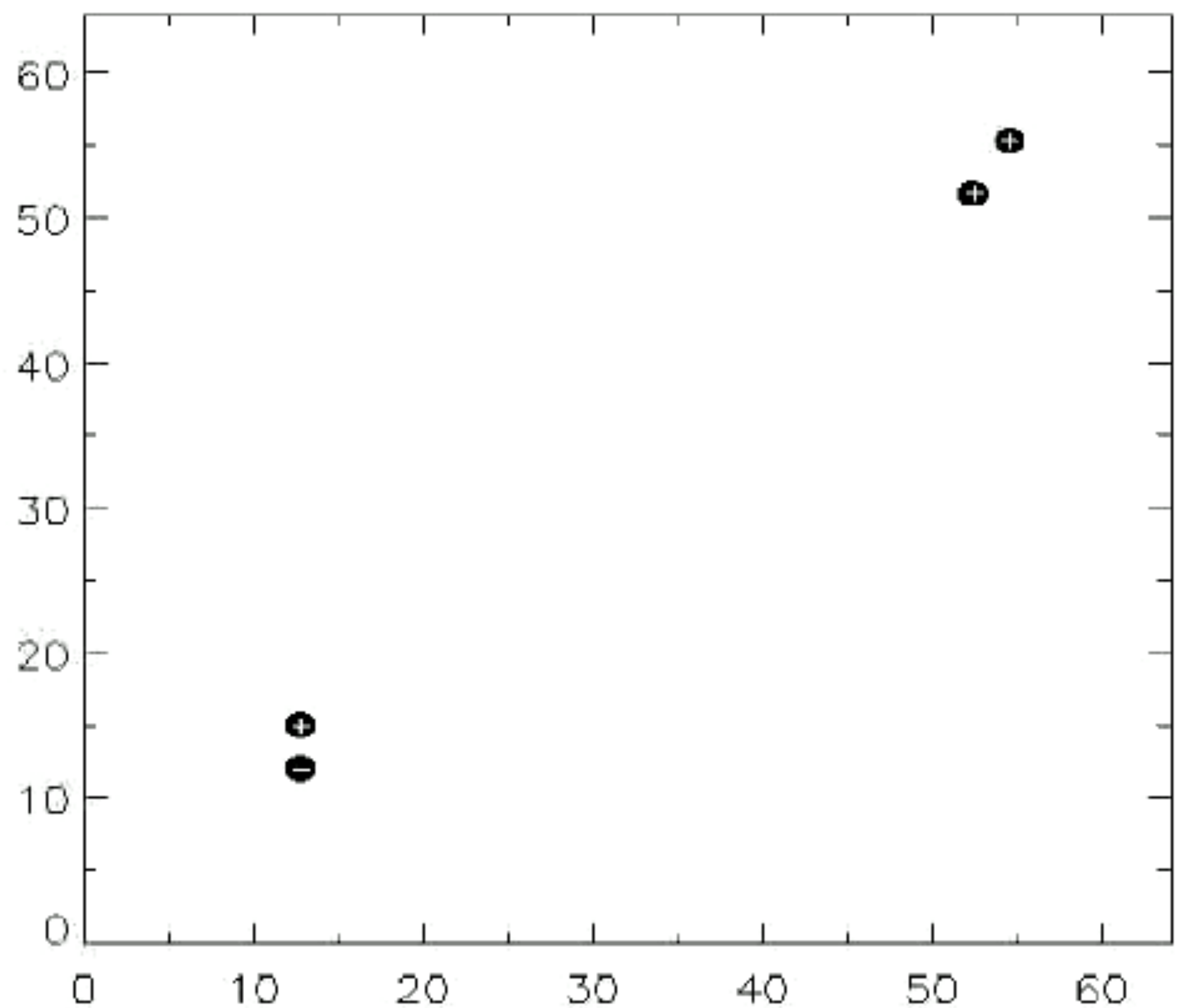


Point (line) vortex:

$$u_\theta : 1/r$$

$$u_r = u_z = 0$$

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi|\mathbf{x}-\mathbf{x}_k|}\left(\hat{z}\times\left(x - x_k\right)\right)$$



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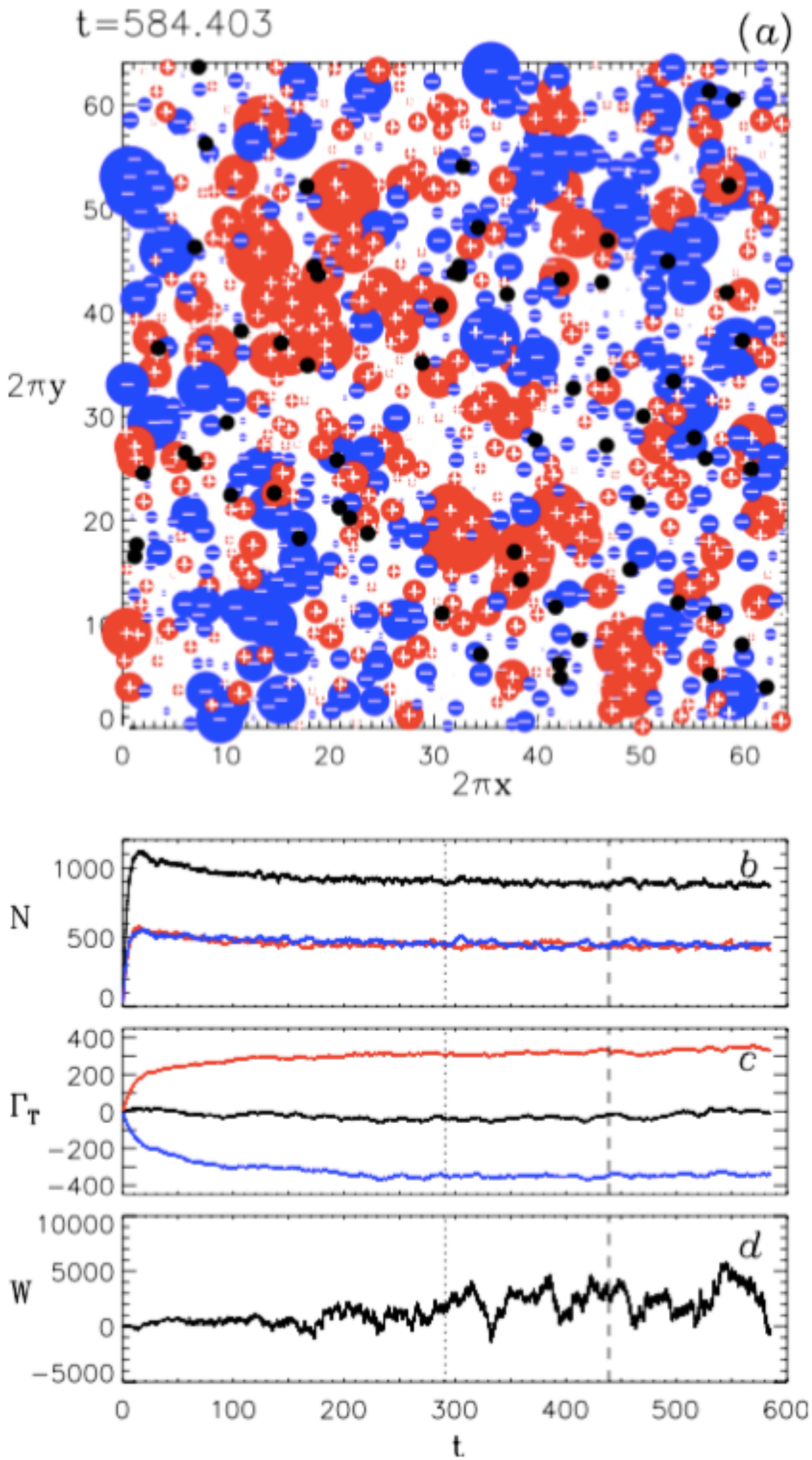
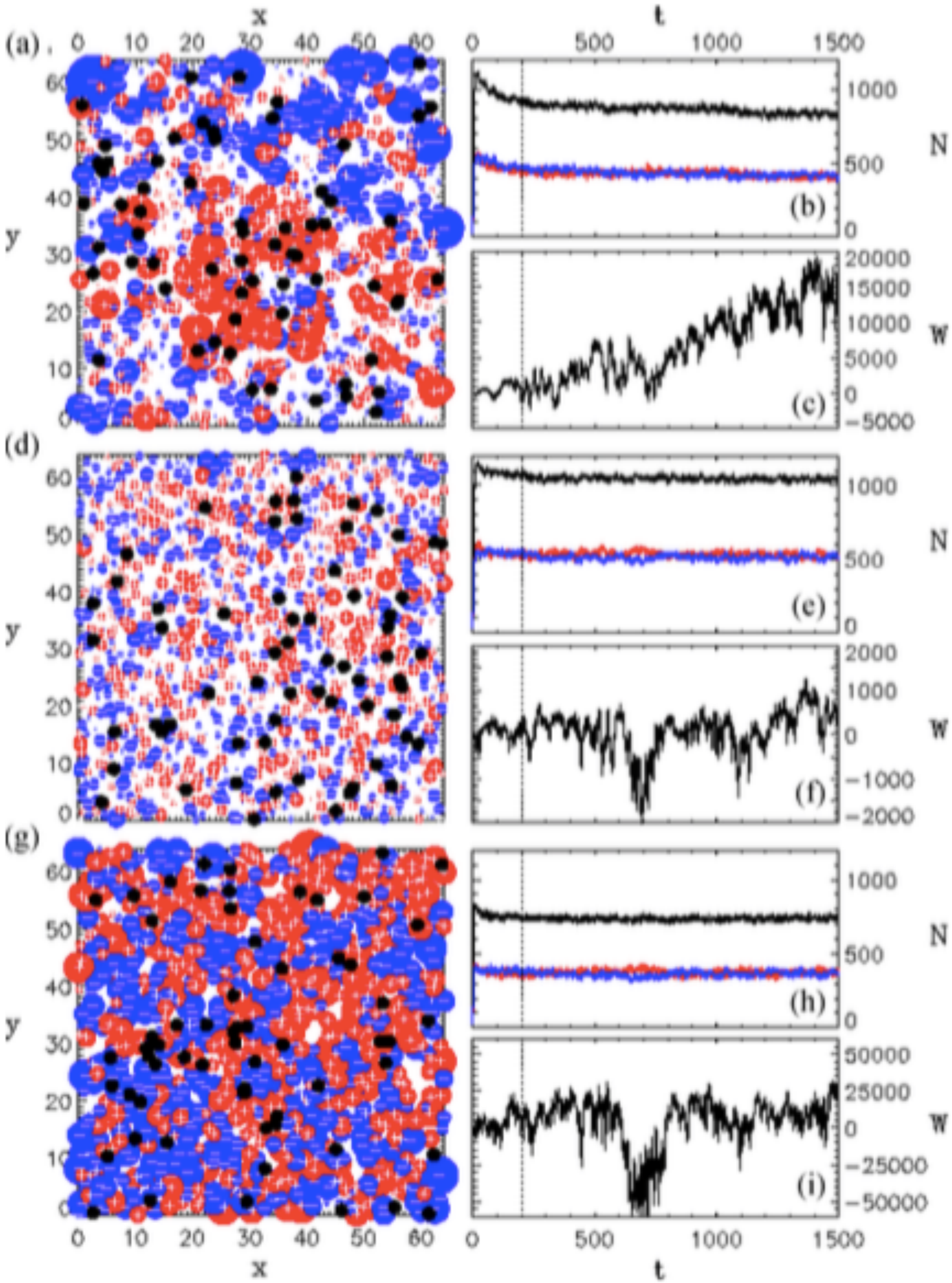


$$\mathbf{u}(\mathbf{x}) = \sum_{1}^{\mathbf{N}} \frac{\mathbf{\Gamma_k}}{\mathbf{2}\pi|\mathbf{x}-\mathbf{x_k}|}\left[\hat{\mathbf{z}} \times (\widehat{\mathbf{x}-\mathbf{x_k}})\right]$$

careful with merger,
long time trends,
injection

$$W = -\frac{1}{4\pi}\sum_{\alpha}^{N}\sum_{\beta}^{N}\Gamma_\alpha\Gamma_\beta \ln(r_{\alpha\beta})$$

Is not a constant because of injection and dissipation in a bounded domain.



Sources and sinks of kinetic energy vs. merger scheme.

(A) circulation conserving : merger of like-sign vortices conserves angular momentum but dissipates energy :

$$(\Gamma_1 + \Gamma_2)^2 > \Gamma_1^2 + \Gamma_2^2$$

while merger of opposite sign vortices dissipates both angular momentum and energy, because

$$(\Gamma_1 + \Gamma_2)^2 < \Gamma_1^2 + \Gamma_2^2$$

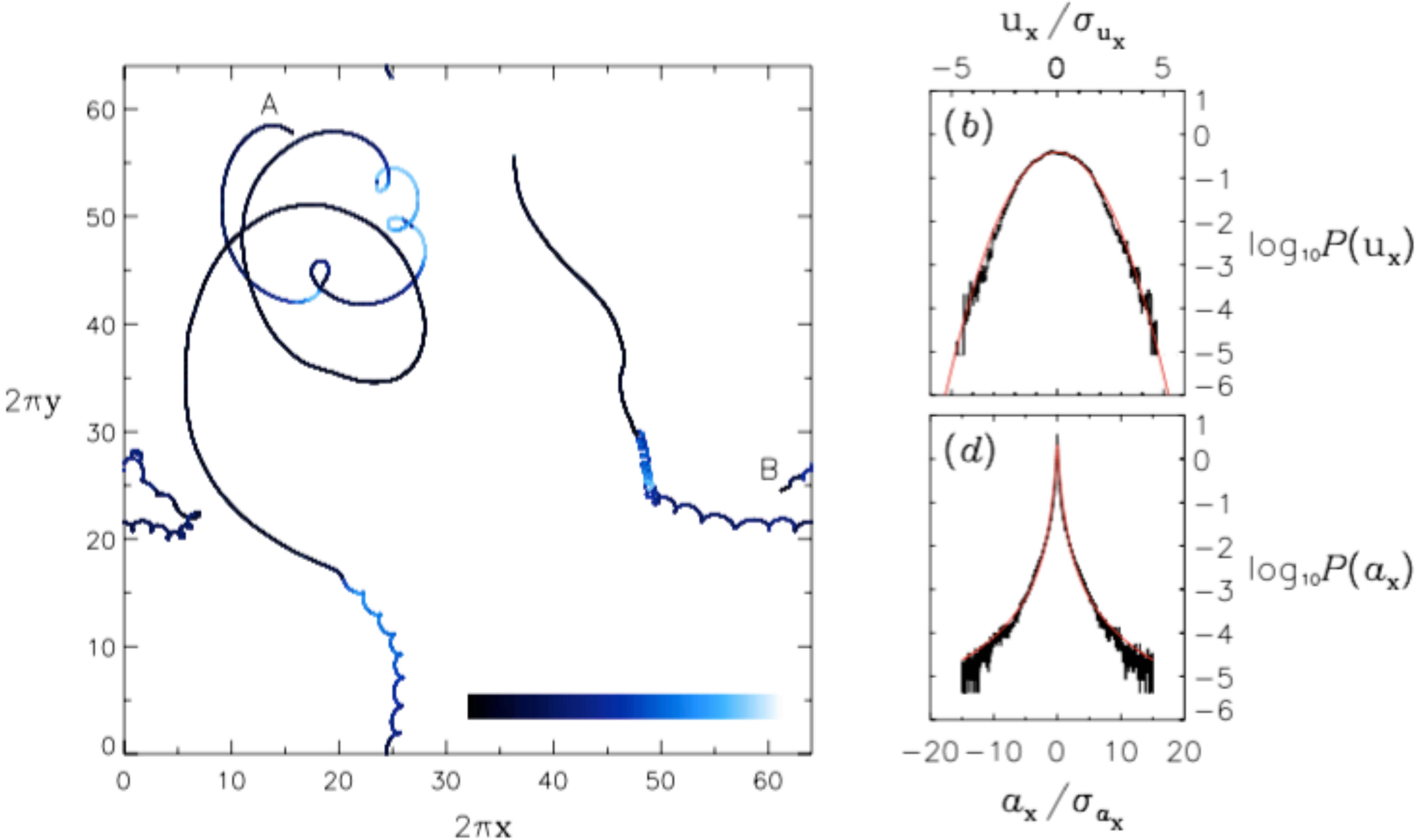
(B) in the signed squared circulation preserving scheme, summing like-sign vortices conserves energy but dissipates angular momentum because

$$\sqrt{\Gamma_1^2 + \Gamma_2^2} < \Gamma_1 + \Gamma_2$$

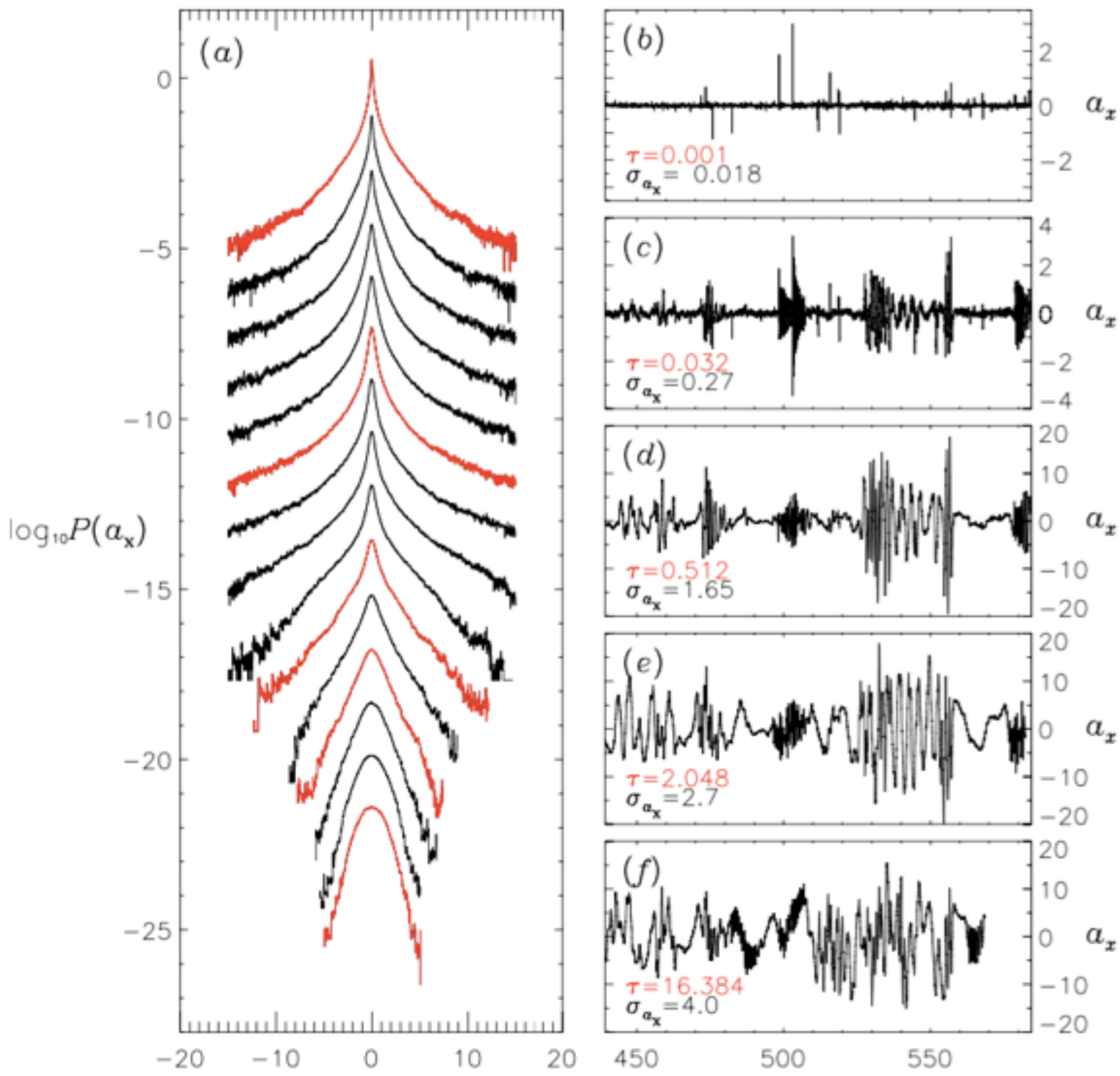
but opposite sign dissipate energy and add angular momentum, since:

$$\mathrm{sgn}\left(\Gamma_1^2 - \Gamma_2^2\right)\sqrt{|\Gamma_1^2 - \Gamma_2^2|} > \Gamma_1 - |\Gamma_2|$$

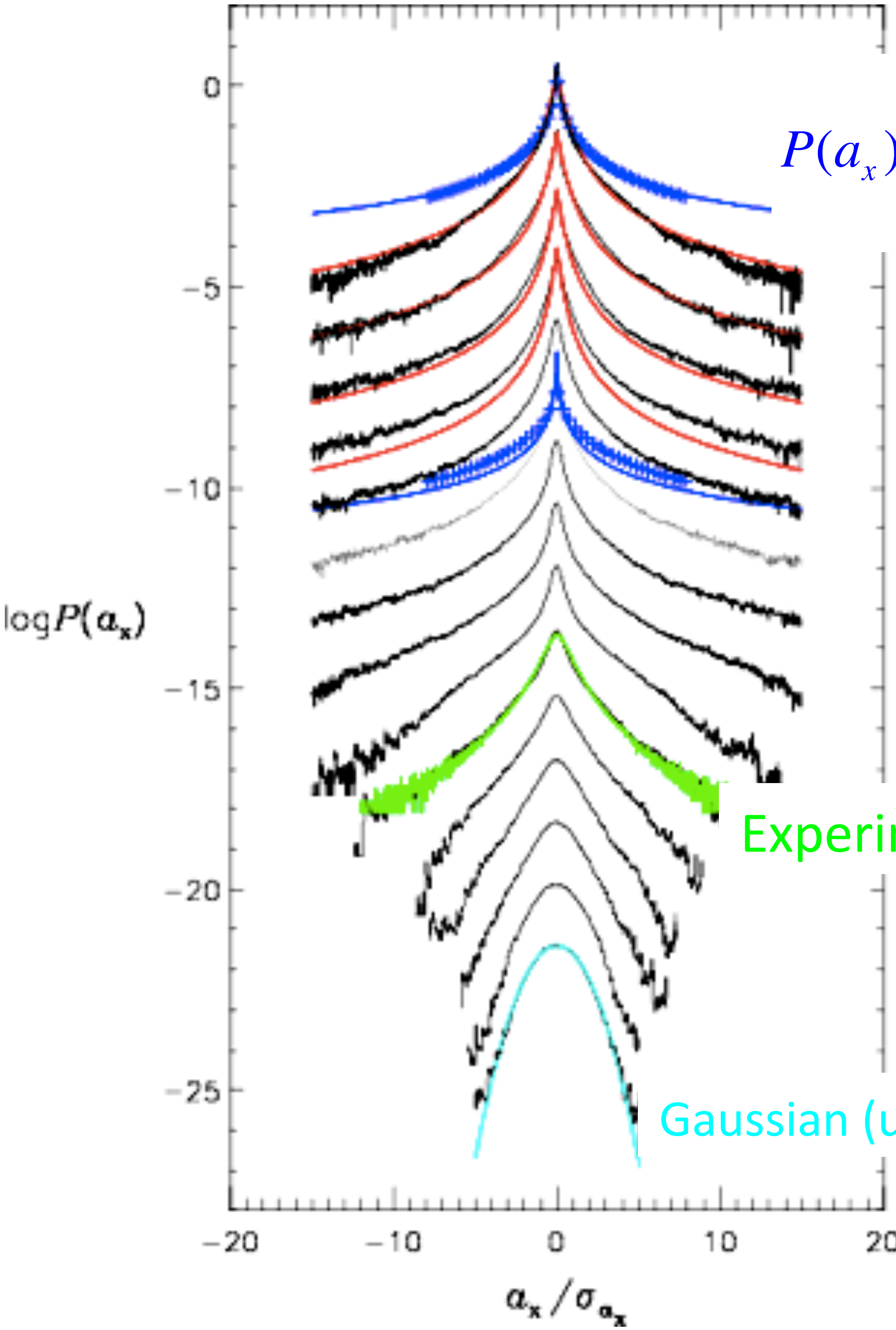
Lagrangian features



and « full » intermittency !



INTERMITTENCY



$$P(a_x) = n\frac{(\sigma^2\tau)^{2/3}}{a_x^{5/3}}\left[a_0 + a_1 n\left(\frac{\sigma^2\tau}{a_x}\right)^{2/3} + a_2 n^2\left(\frac{\sigma^2\tau}{a_x}\right)^{4/3} + \dots\right]$$

$$P(u_x) = \frac{2}{\pi^3}\frac{1}{\sqrt{u_x^2 + 2\pi n\sigma^2}}\left[K\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right) - E\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right)\right]$$

with NO fitting parameters.

Experimental data

Gaussian (uncorrelated)

NB: analytics with
- random sampling of a single vortex
- random distribution of nearest neighbor
- … with random amplitude distribution

Pair dispersion

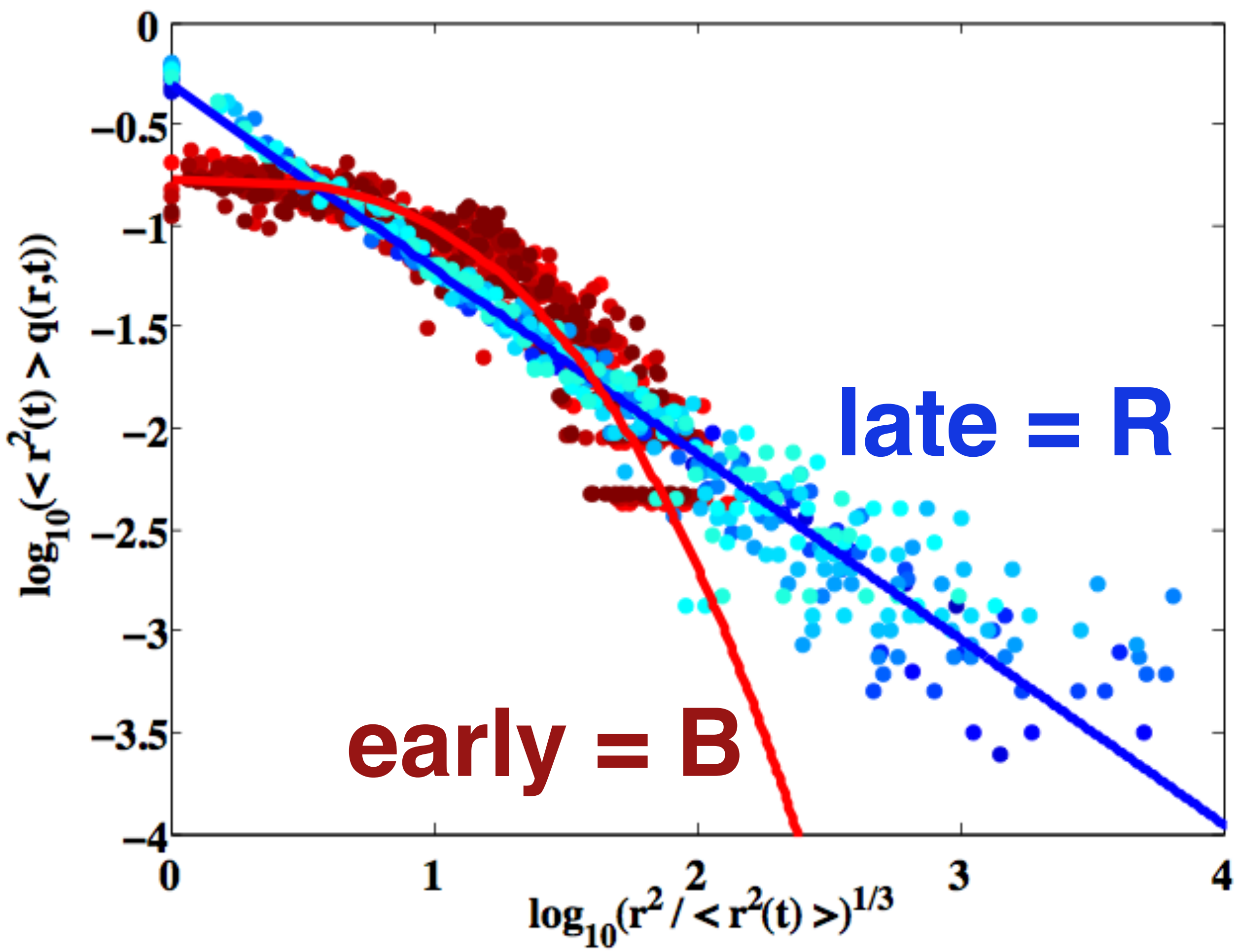
Scaling expectations

- Very short times : exponential separation following the larger Lyapunov exponent,
- Very long times : diffusion : $\langle r^2(t) \rangle \propto t$
- Intermediate : Richardson - Obukhov : $\langle r^2(t) \rangle \propto \langle \epsilon \rangle t^3$
- Bachelor correction : $\langle r^2(t) \rangle \propto (\langle \epsilon \rangle r_0)^{2/3} t^2$

scaling …

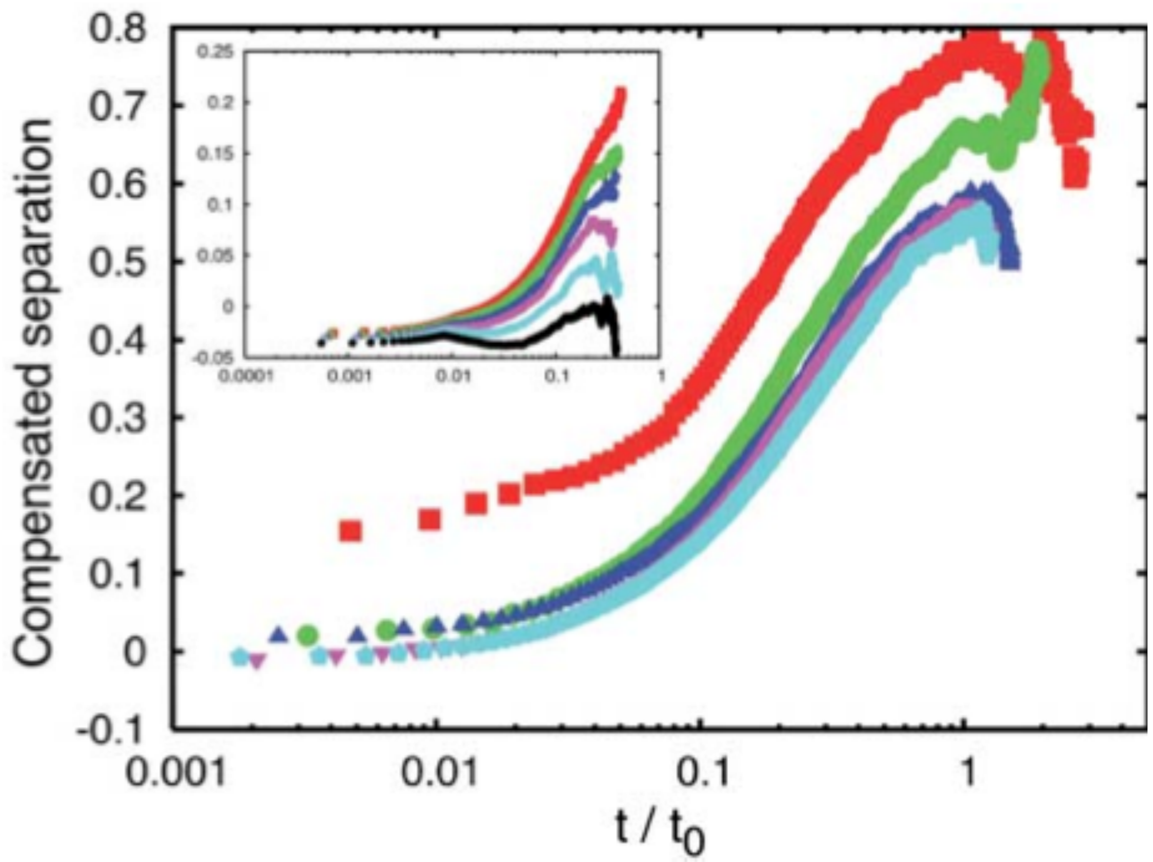
- Neighbor distance function
- $$q_R(r,t) = \frac{429}{70}\sqrt{\frac{143}{2}}(\pi\langle r^2\rangle)^{-3/2}\exp\left[-\left(\frac{1287 r^2}{8\langle r^2\rangle}\right)^{1/3}\right]$$

$$q_B(r,t) = \left(\frac{2\pi}{3}\langle r^2\rangle\right)^{-3/2}\exp\left[-\frac{3}{2}\frac{r^2}{\langle r^2\rangle}\right]$$

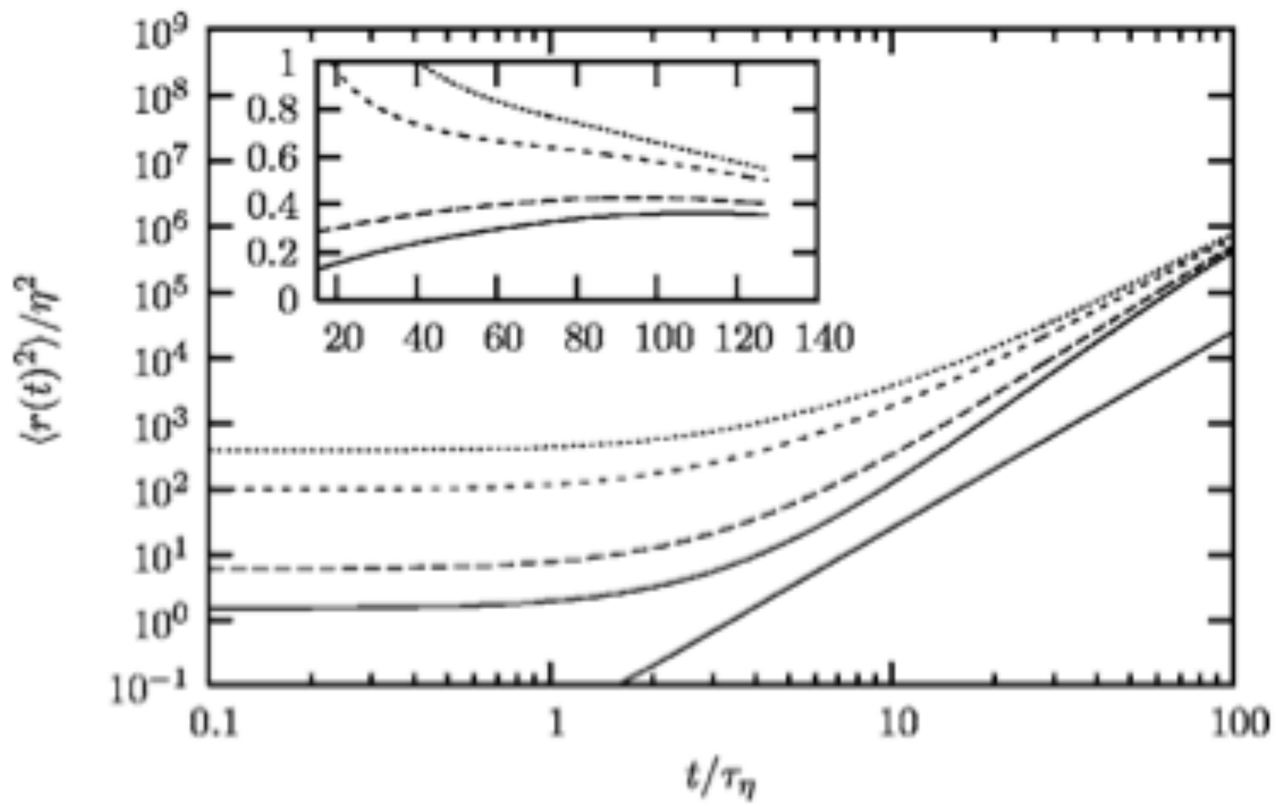


… or nor scaling!

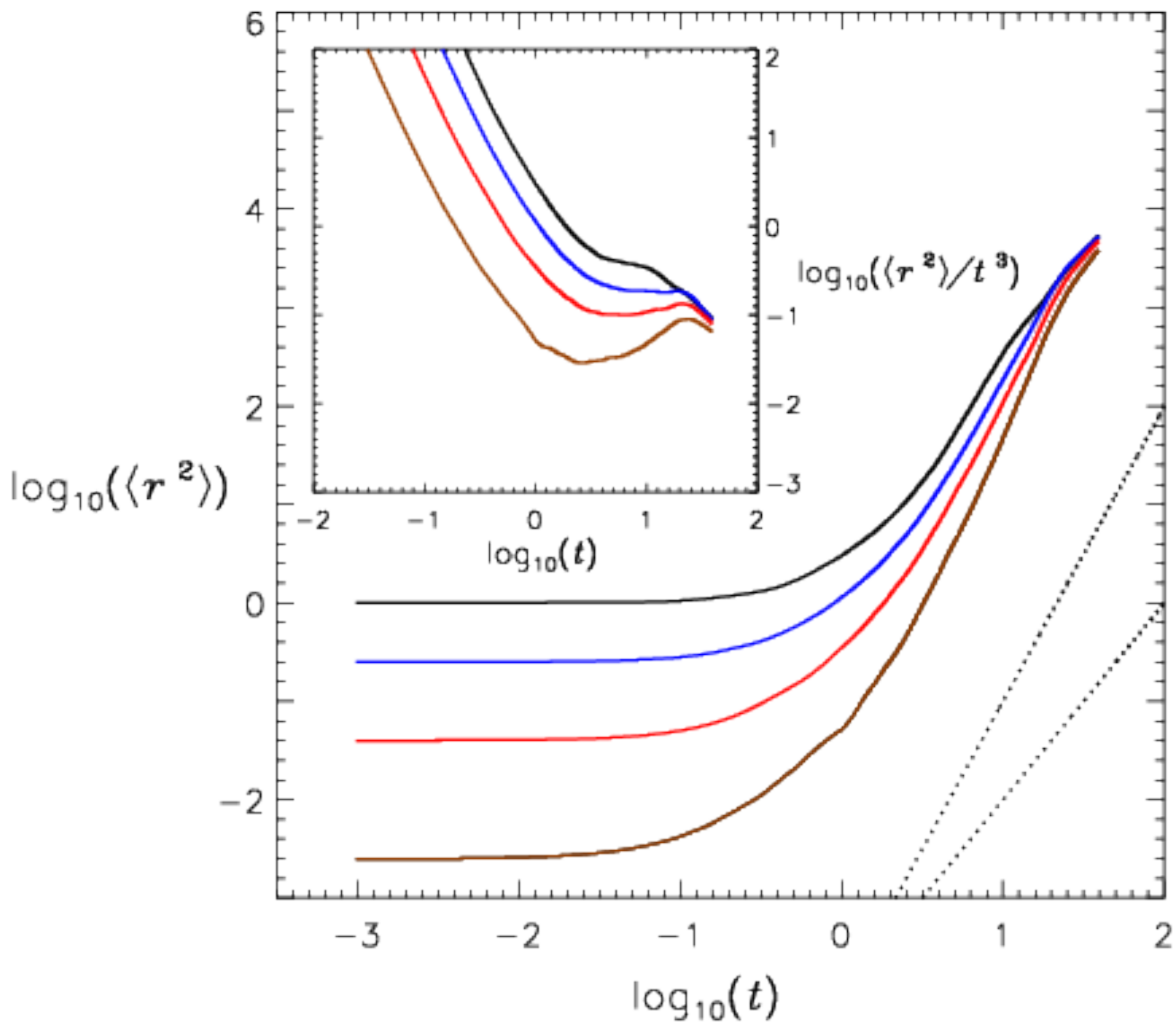
Bourgoin et al. 2006



Biferale et al. 2006

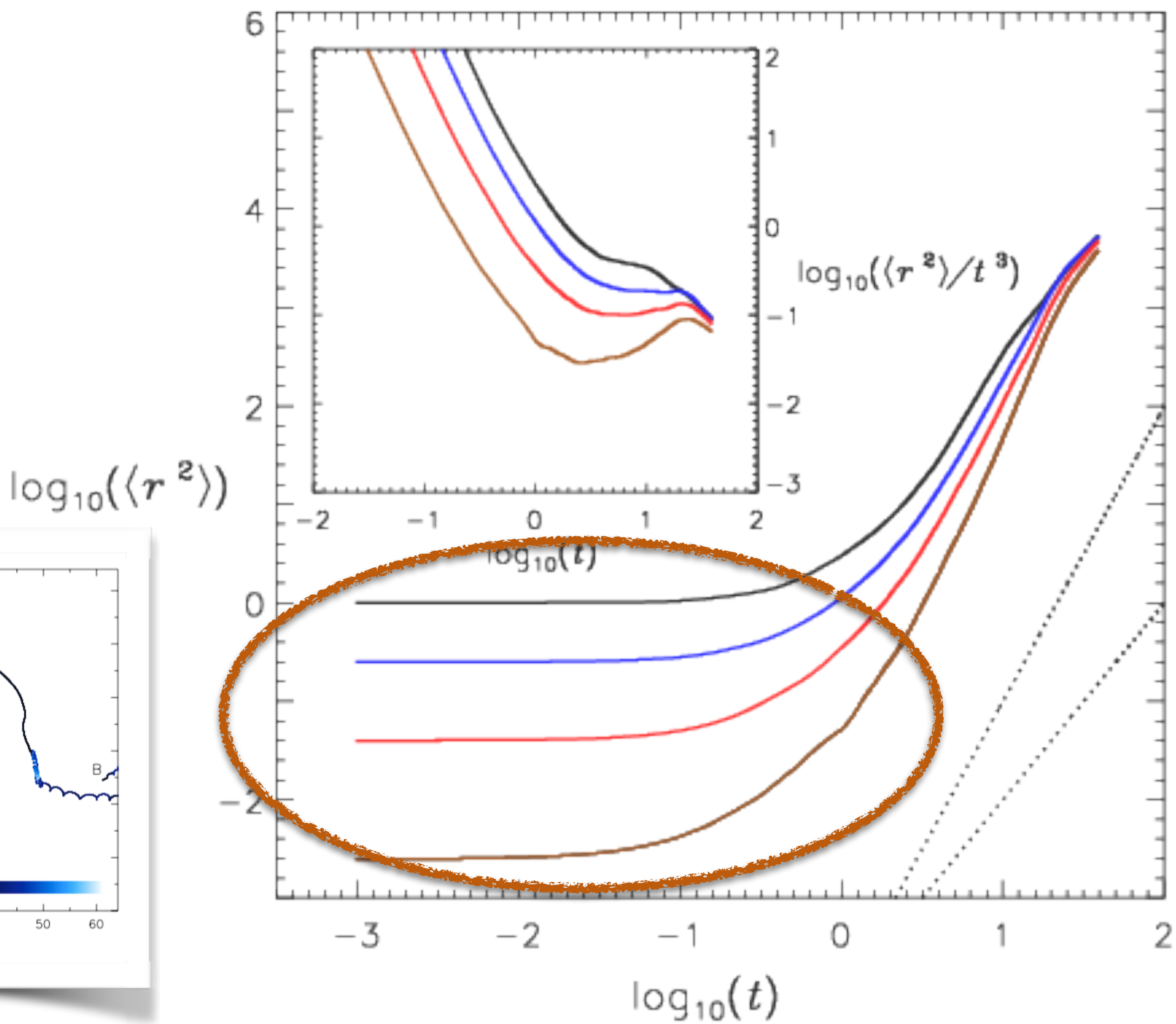


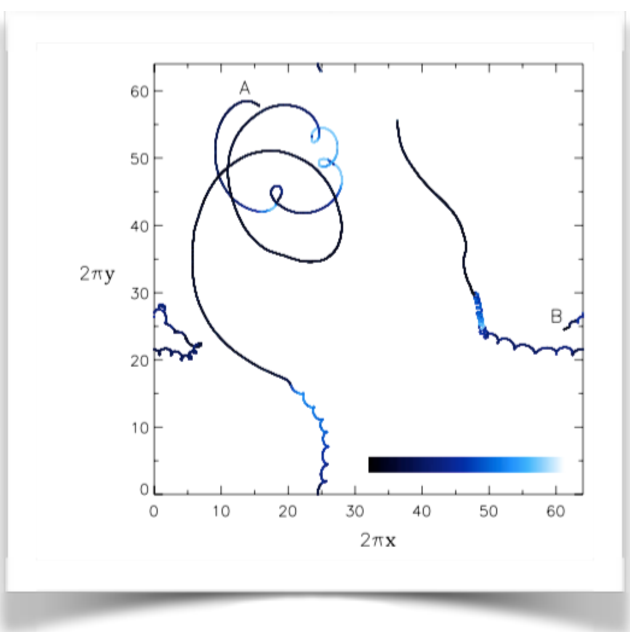
… or nor scaling!



scaling-wise, the point vortex model is as « convincing » as other exp or num !

... or nor scaling!





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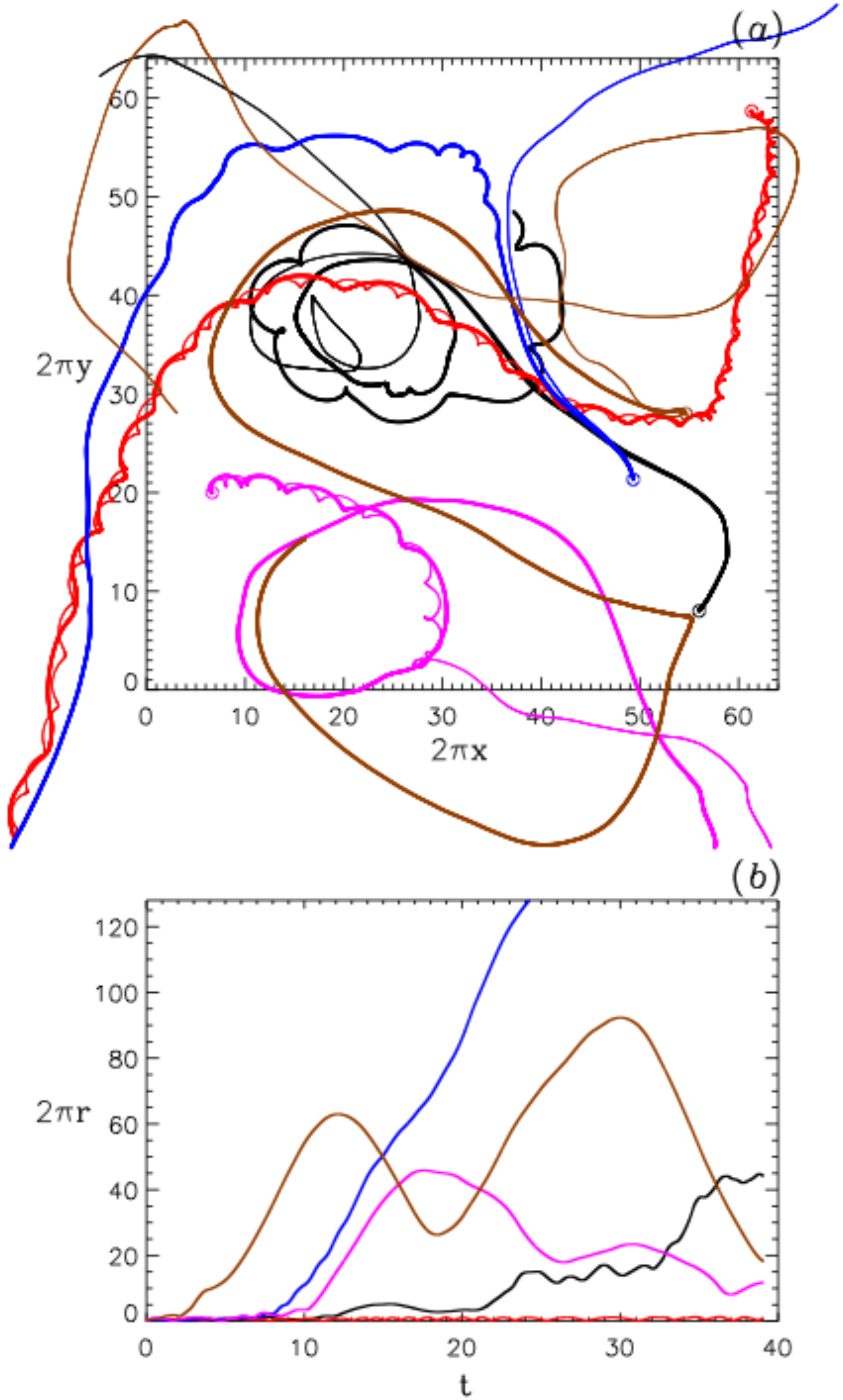
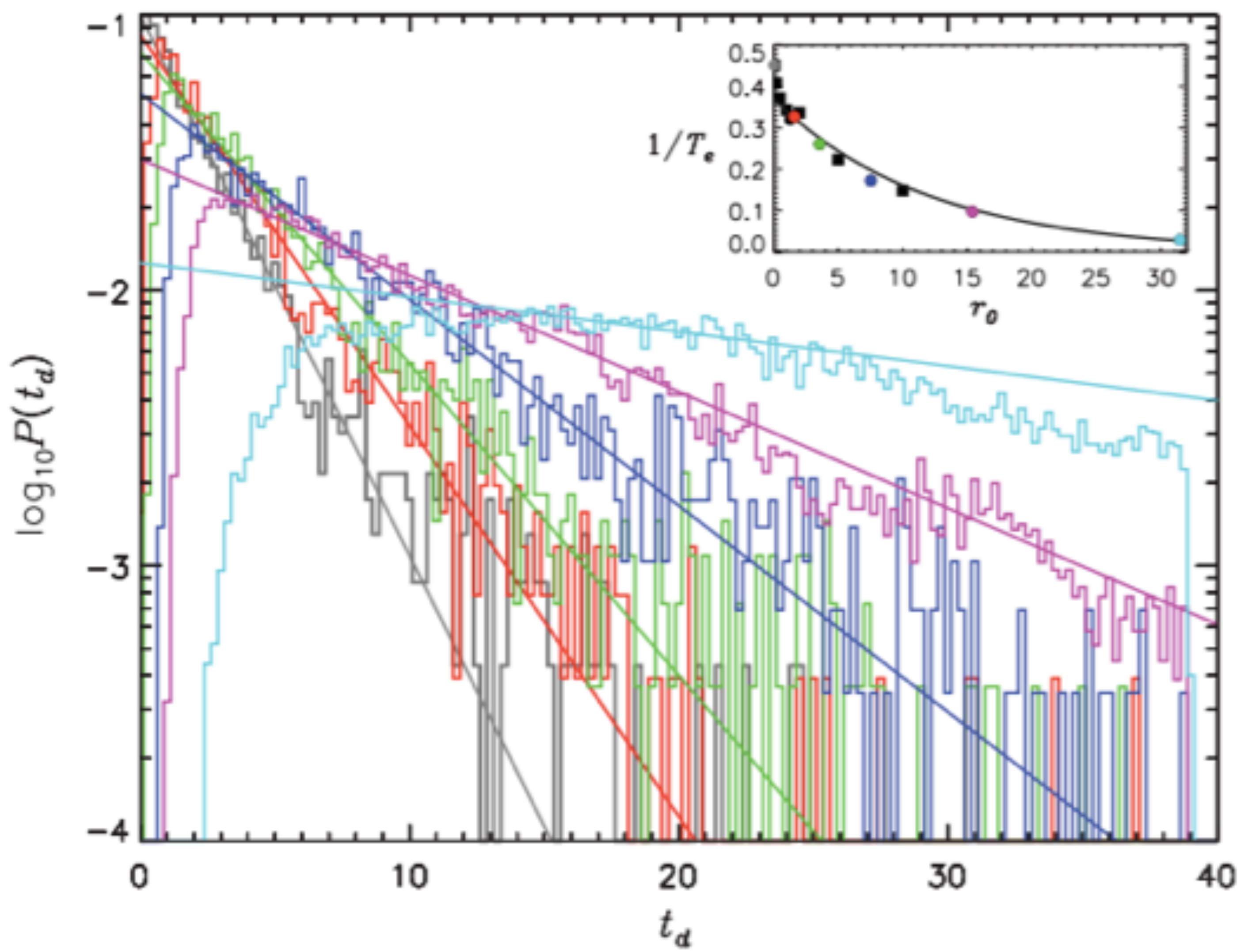

(a)
60
50
40
30
20
10
0
2πy
0
10
20
30
40
50
60
2πx
(b)
120
100
80
60
40
20
0
2πr
0
10
20
30
40
t

separation

Distribution of times required to double the initial separation.
r0 = 0.05, 1.29, 3.57, 7.53, 15.4, and 31.4

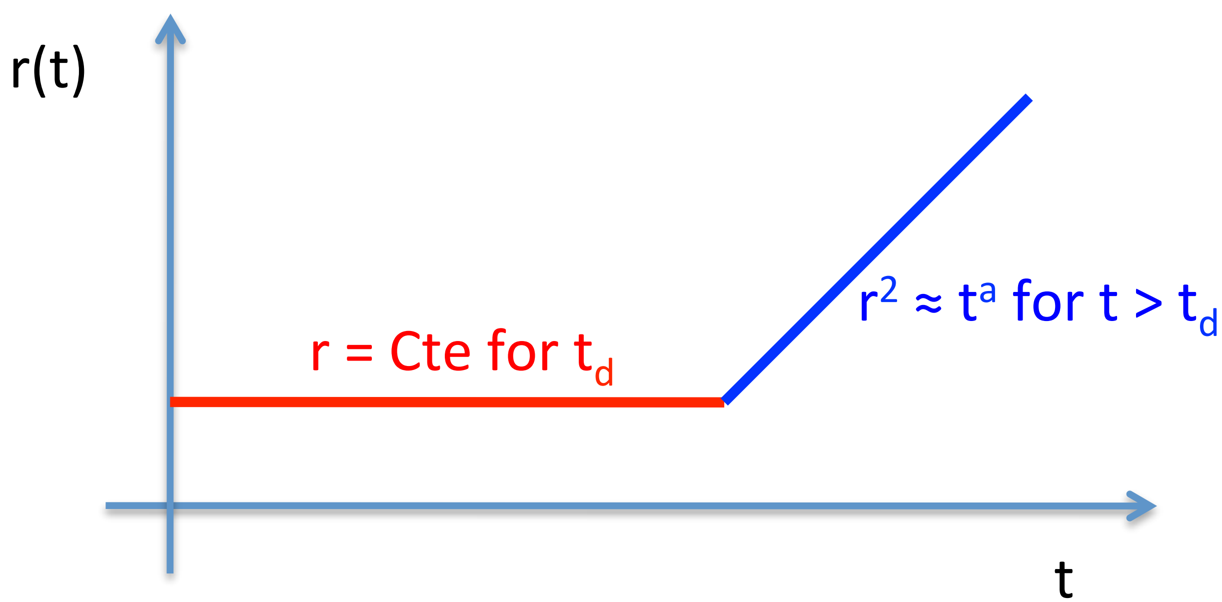
That is, even if many pairs separate right away, a significant fraction can remain bound for times comparable to the large scale eddy turnover time.

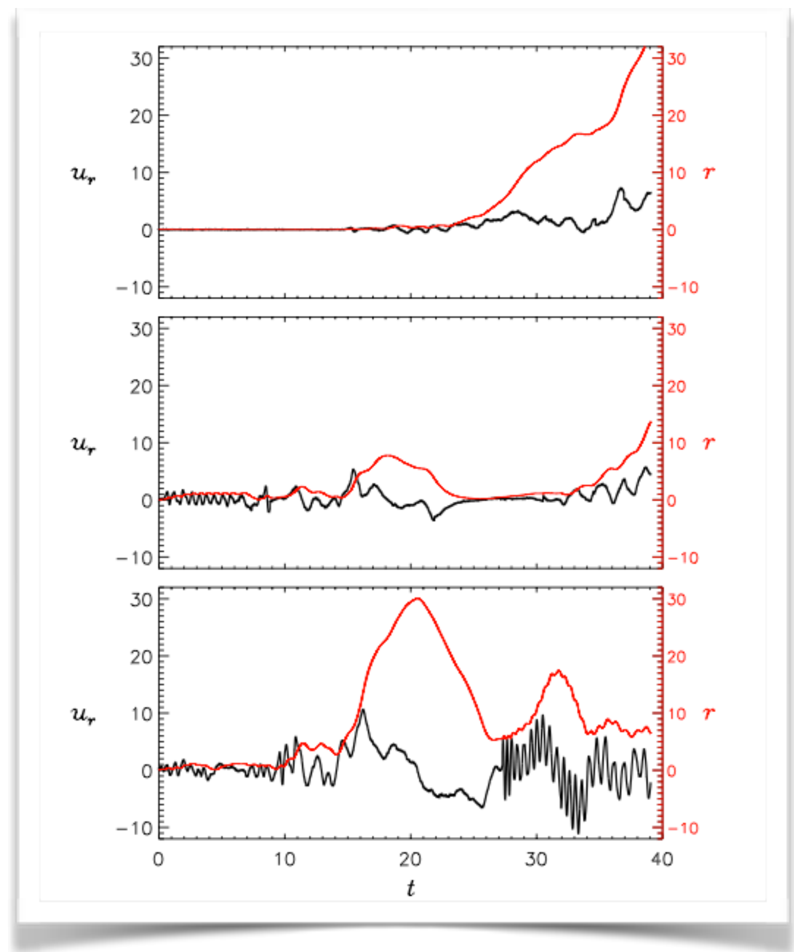


Model of the model

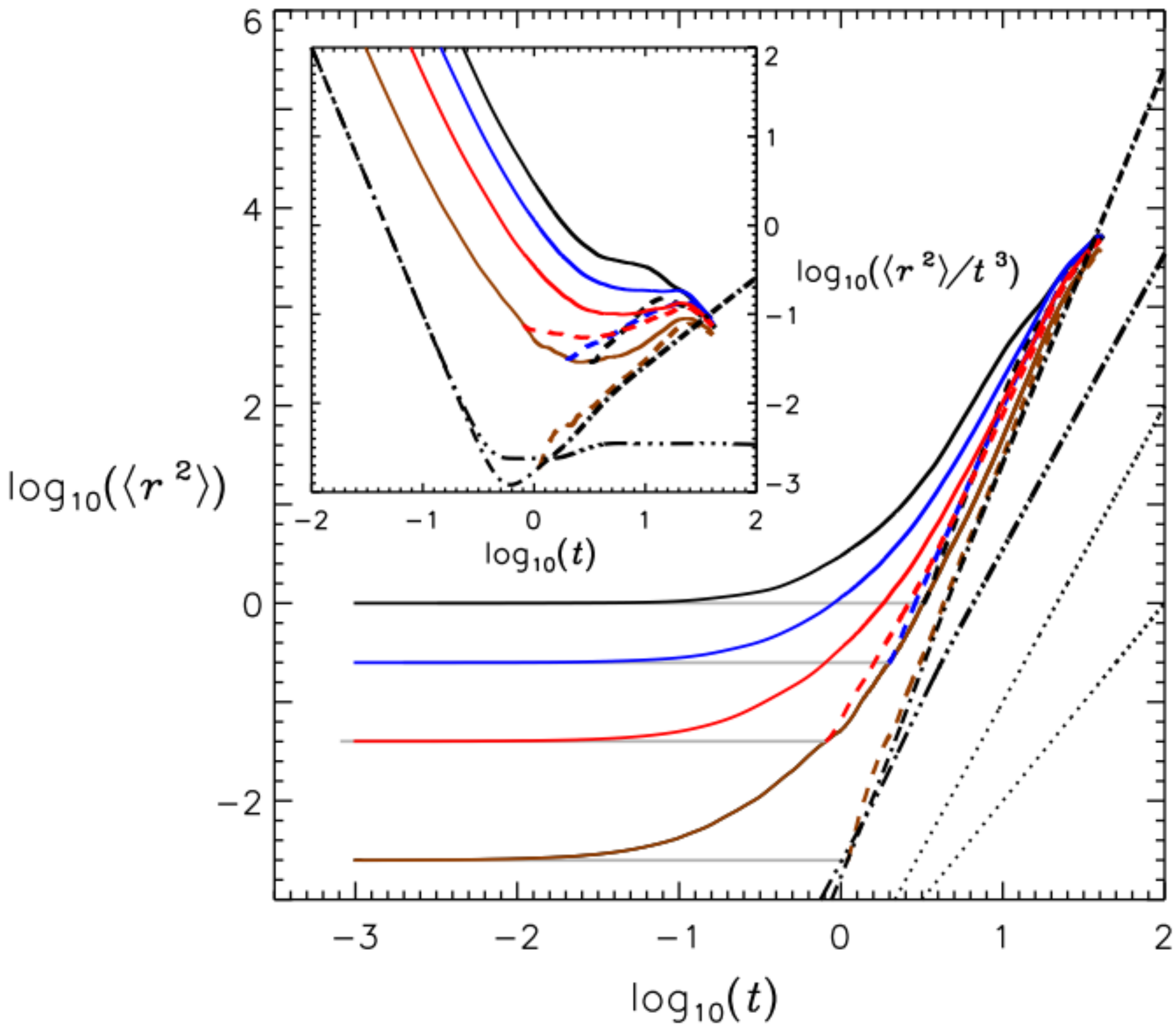
- Step 1 : particles retain their initial separation r0 for a time td (uniform)

- Step 2 : they separate algebraically $r^2 = r_0^2 + (t - t_d)^\alpha$



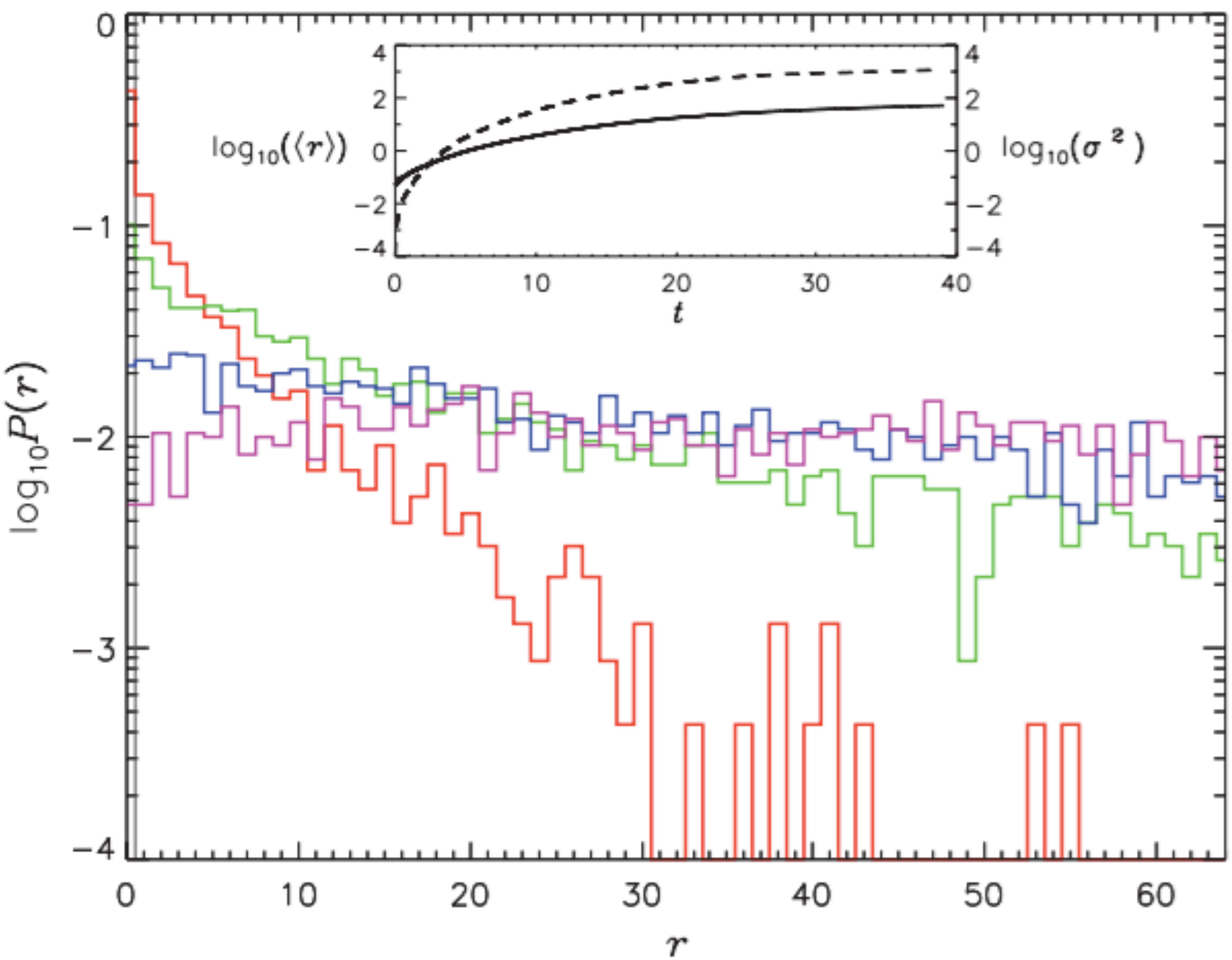


Separation



6
4
2
0
−2
log$_{10}$(⟨r^2⟩)
−3
−2
−1
0
1
2
log$_{10}$(t)
2
1
0
−1
−2
−3
log$_{10}$(⟨r^2⟩/t^3)
−2
−1
0
1
2
log$_{10}$(t)

Distribution of separations



Dynamical picture

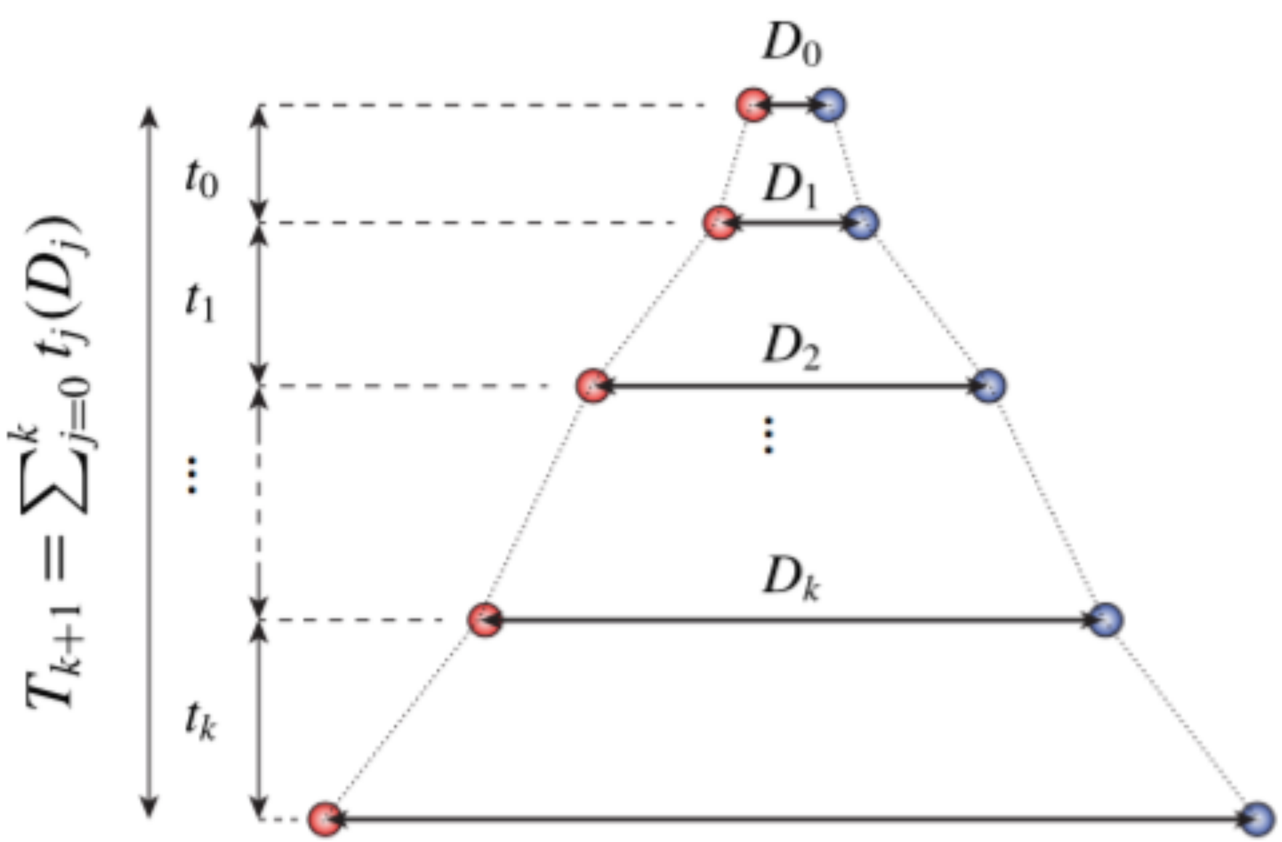
- $r(t)$ evolves as a sequence of expansion and stall phases,
- stall phases (in the model) come from coherent eddies,
- at all separations, a pair has a uniform probability of being stalled any amount of time, i.e. it has the same probability of separating right away or being stuck together « for ever »,
- here, scaling is subdominant because of the leading influence of coherent structures / trapping !

Statistical picture

J. Fluid Mech. (2015), vol. 772, pp. 678–704.
doi:10.1017/jfm.2015.206

Turbulent pair dispersion as a ballistic cascade phenomenology

Mickaël Bourgoin†

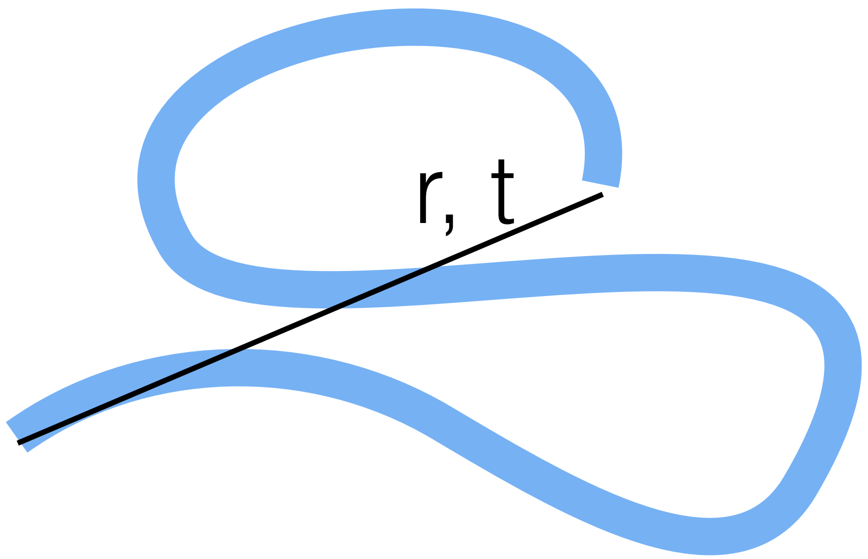


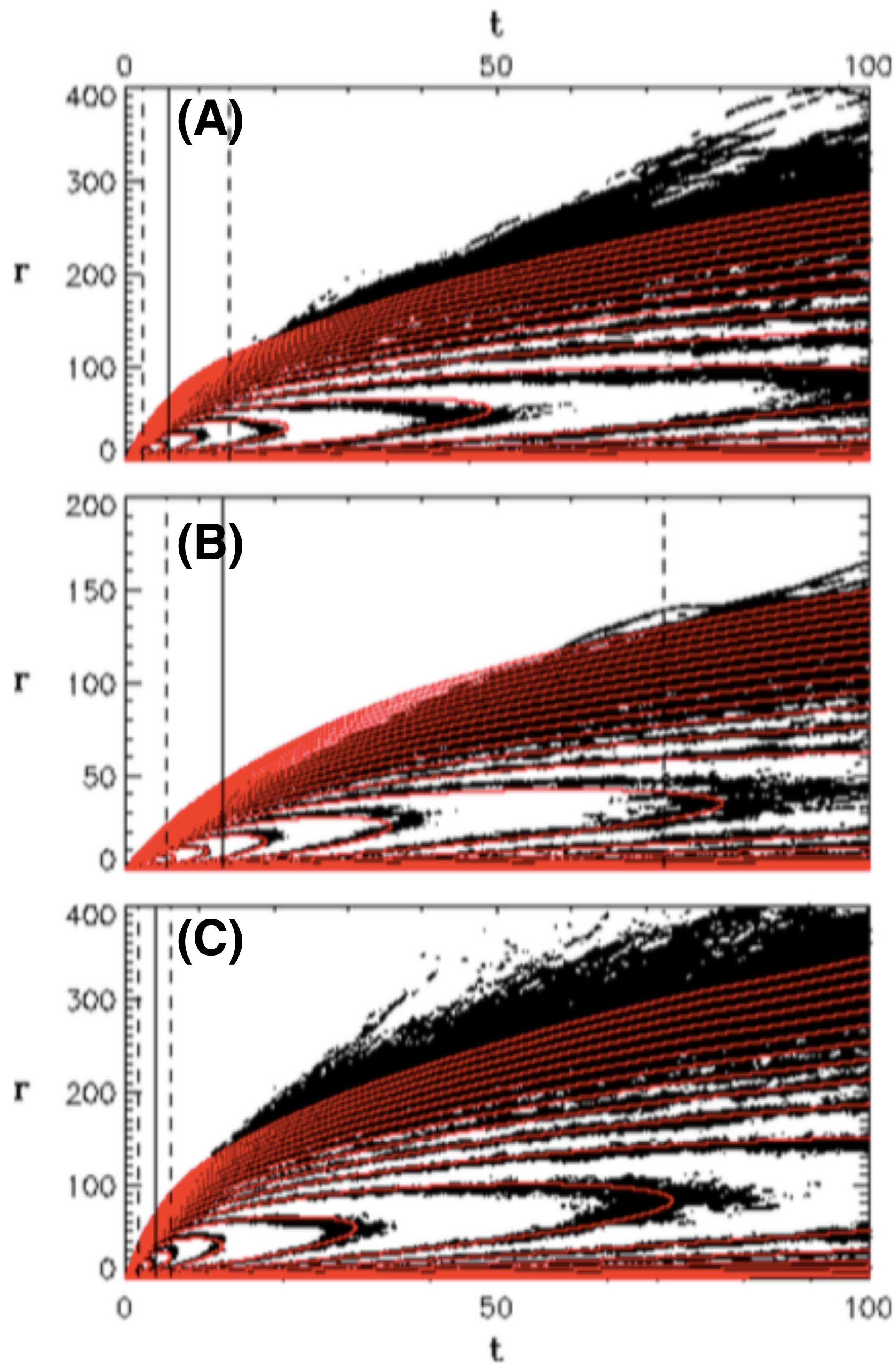
$$D_{k+1}^2 = D_k^2 + S_2(D_k)t_k^2(D_k)$$

Scalar concentration

$$\langle c(\mathbf{x},t))\rangle = \int\int P(\mathbf{x},t|\mathbf{x}',t')S(\mathbf{x}',t')d\mathbf{x}'dt'$$

- direct determination of P(x,t | x',t') is overly ambitious
- isotropy : P(x,t' | x',t '') = P(r,t) / 2πr r=|x-x'|, t = t'-t''
- so that r is the Eulerian distance traveled by a Lagrangian tracer during a time interval t



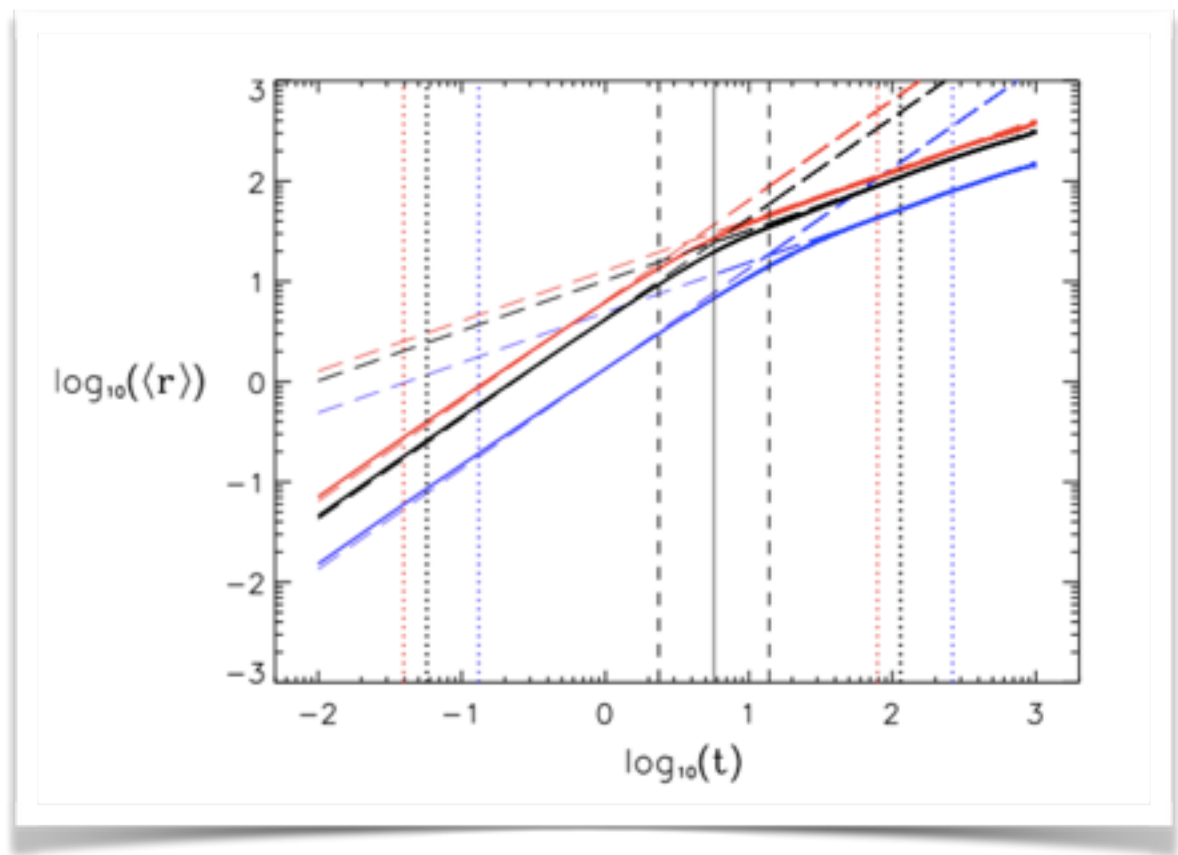


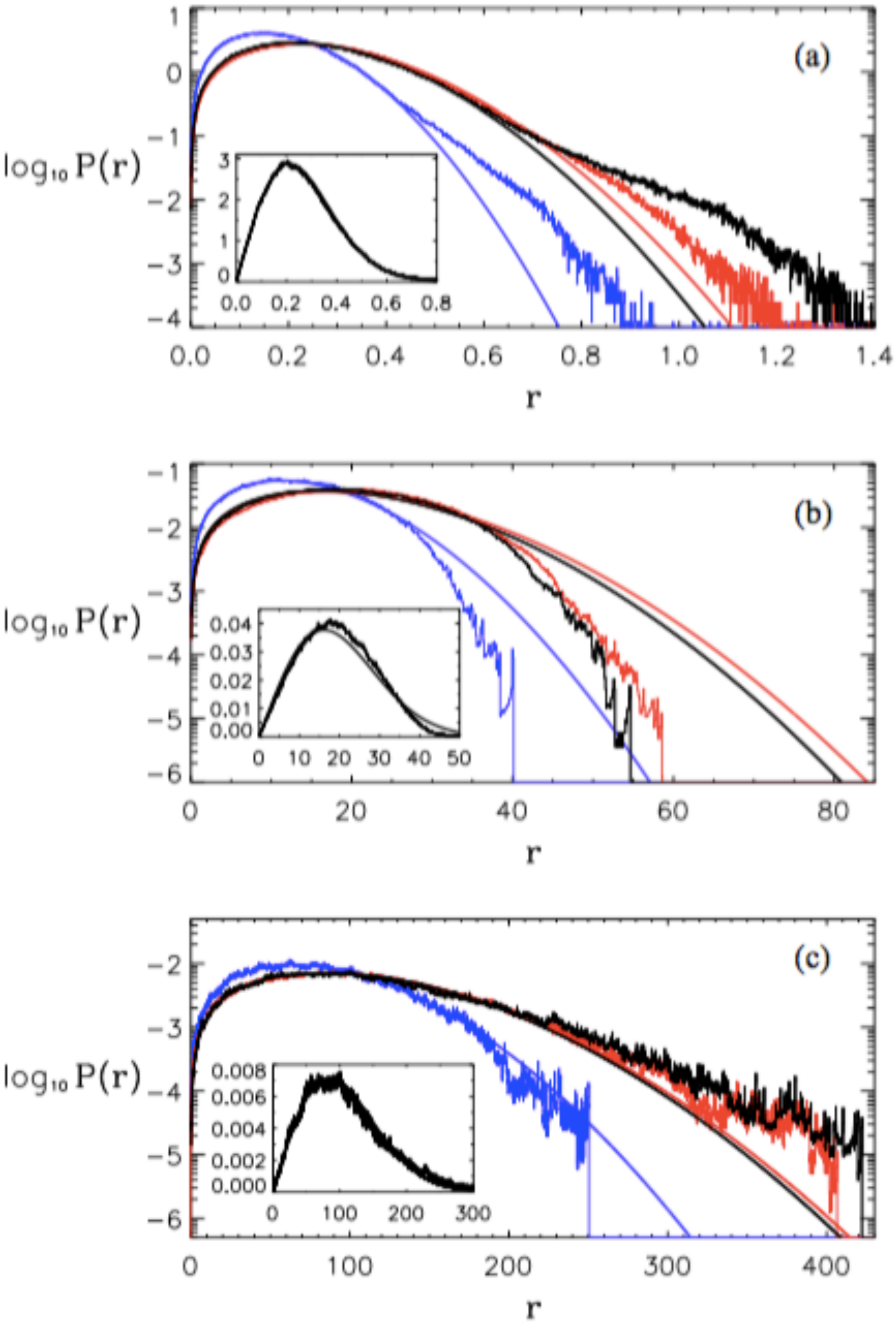
Starting two hundred time units into each of the simulations.

RED : Rayleigh distribution

$$P(r,t) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

of a 2D random walk,





The probability of long distance transport is enhanced over the best fit two-dimensional random walk Rayleigh distribution at very short and, in some cases, very long times [(a) and (c)] but suppressed over intermediate (inertial range) time scales (b).

Structures ?

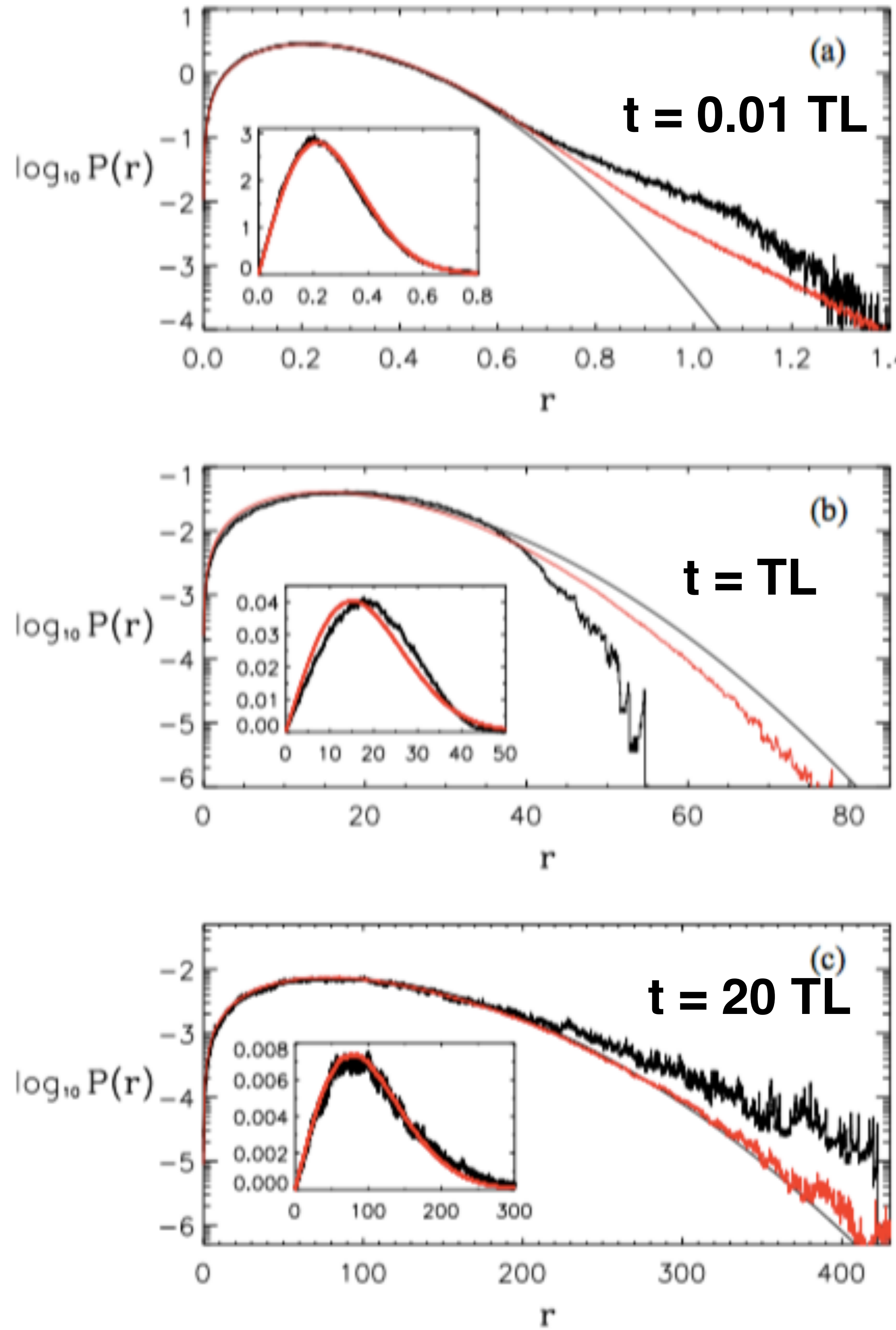
Construct an artificial times series with the same auto and cross correlation coefficients as the vortex flow :

$$\tilde{u}_x = \sqrt{\tilde{U}_L} \exp i\delta_x$$

with uniformly distributed random phases (tilde = FT), and

$$\tilde{u}_y = \tilde{u}_x \exp -i\delta_y \quad \delta_y = -i \ln \left(\frac{\tilde{C}_L}{\tilde{U}_L} \right)$$

simulation (A)



Eddy constrained random walk

Circular motion around an eddy,

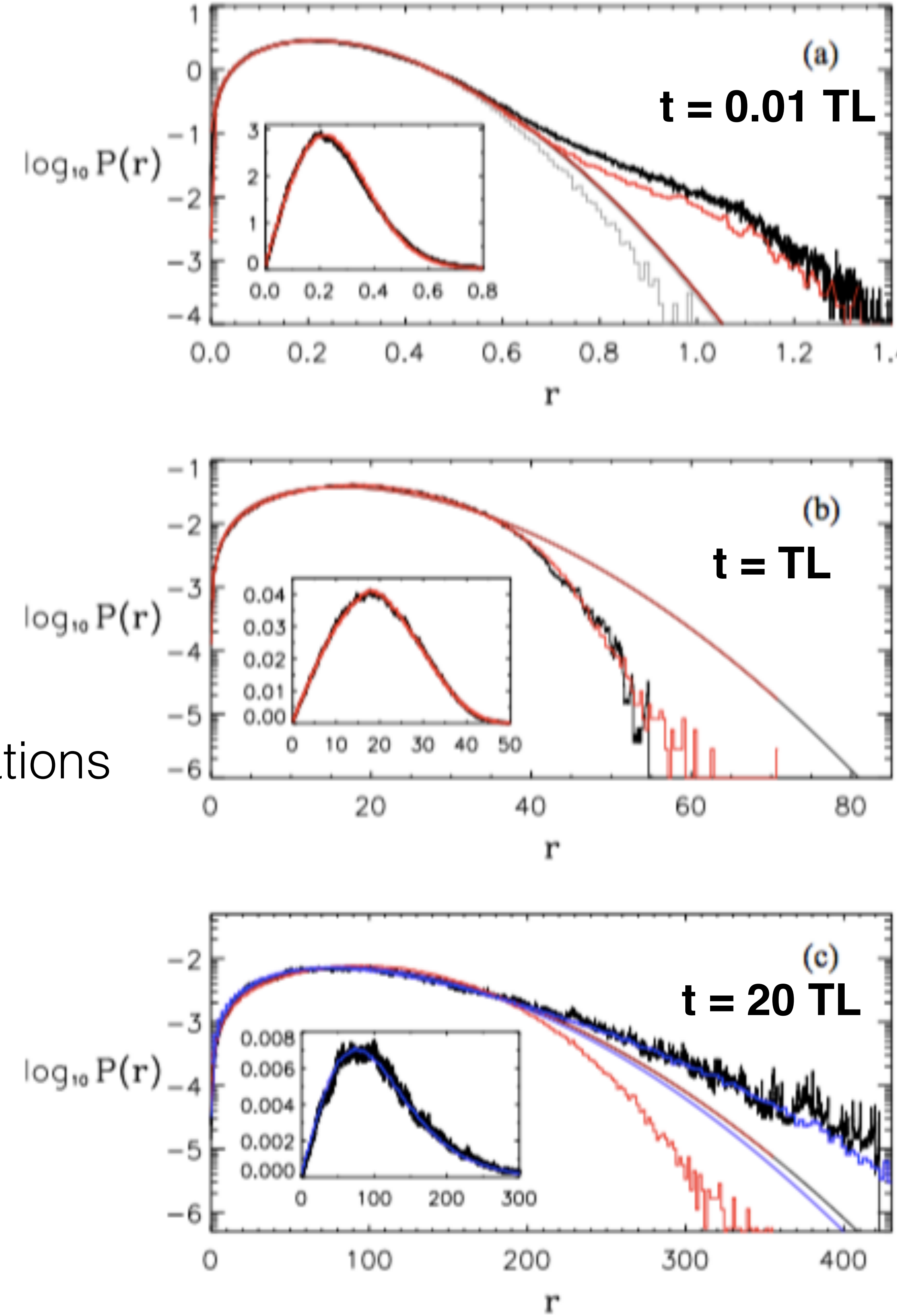
$$dr = 2r_t|\sin\theta| \qquad r_t\theta = U_L t_t$$

random sequence of such events,
and P(r,t) is build from

$$P(r_t)\ ,\ P(t_t)\ ,\ P(U_L)$$

with random (uniform) direction, and
$P(t_t)$ from observation : uniform between 0 and TL
$P(r_t)$ from « turbulence » phenomenology : $\propto r_t^{4/3}$
$P(U_L)$ from Lagrangian observation.

simulation (A)

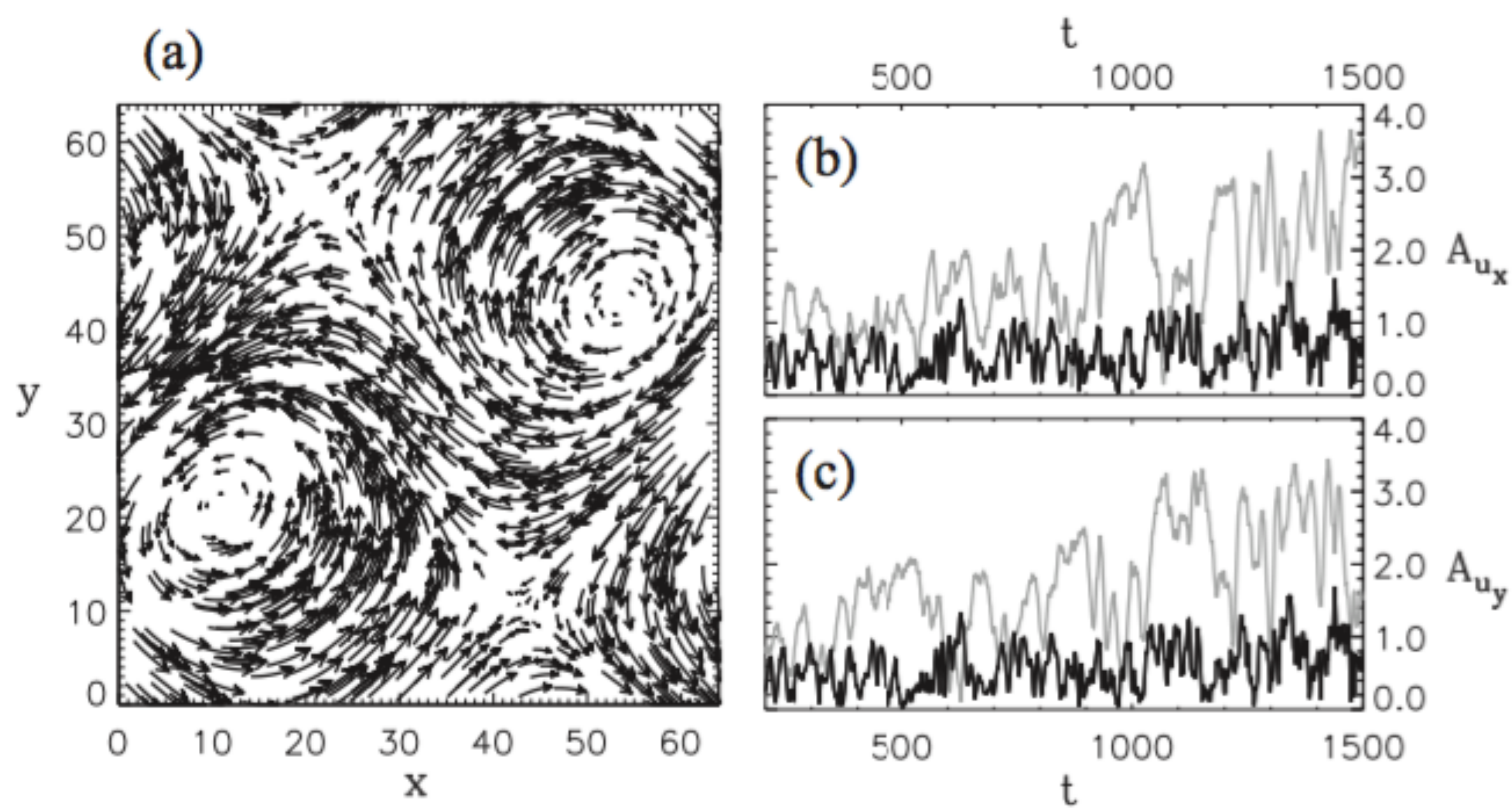


BLACK : P(r, fixed t) for run (A)

RED : constrained random walk with random orientations

BLUE : large scale component fluctuations are added

Low wavenumber contributions



Add, at the position of the Lagrangian parcel, the Eulerian velocity at time t of the lowest order modes, i.e. large scale motions which still evolve in time at large scale.

[0,1] and [1,0]

conclusions

- It requires only the observed amplitude evolution of the lowest wave number modes (the mean and the lowest harmonic) and measurable statistics of the smaller scale flows (used in a constrained eddy to eddy random walk) to reproduce the scalar transport probabilistic impulse response function.

- NB : 3D : in progress : cf. Pablo Minini : on time scales set with collaborators very slow response time.

thank you